ANALOG SOLUTION OF CENTRAL FORCE PROBLEM

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ABSTRACT

Electronic analog computers have been used extensively for the solution and display of many dynamics problems. The majority of the problems worked with have been those involving linear differential equations with constant coefficients. In cases involving non-linear differential equations fewer solutions have been developed. This has been due mainly to the need of using non-linear elements in the computer circuits when setting up the solutions.

One such problem, that of a mass moving under the action of a first power central force, is treated in some detail. The differential equation is derived, the problem is scaled, and the circuitry developed. Solutions obtained by the use of the electronic analog computer are displayed and compared with solutions obtained by numerical methods and errors and their sources are discussed. Finally there is an overall evaluation of the usefulness of analog computers to this sort of problem. In an appendix, a second practical dynamics problem is discussed, but a solution was not obtained due to lack of time available.

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LIST OF ILLUSTRATIONS

TABLE OF SYMBOLS AND ABBREVIATIONS

A, B, C	(without subscripts) Constants in differential equation
C_i	Capacitor (i = f, 1, 2, 3, ...)
M	Megohm
R	(without subscript) Voltage representing radius r
R_i	Resistor (i = f, 1, 2, 3, ...)
X	A capital representing the voltage equivalant of a variable x
Z	Output voltage of division circuit
a	Coefficient potentiometer value
e_i	Input voltage to an operational amplifier
e_o	Output voltage of an operational amplifier
f	(subscript) Element in feedback loop
r	Radius
t_c	Computer time
t_p	Problem time
α_i	Scaling factor (i = 1, 2, 3, ...)
θ	Angle of rotation
ω_i	Input voltage coefficient (i = 1, 2, 3, ...)
	Capacitor
	Resistor
	Operational amplifier
	Coefficient potentiometer
	Function Multiplier

1. Introduction

This thesis presents the electronic analog solution to a non-linear dynamics problem which leads to the differential equation $\ddot{X} = \frac{A}{X^3} + BX + C$[1]. An example problem is taken and the steps in reducing it to a form suitable for an electronic analog computer, hereafter referred to as an analog computer, are shown. The results are then compared with two solutions obtained by numerical methods. In Appendix III an equation of the form; $\dot{X}^2 = AX^3 + BX^2 + CX + D$ is discussed and the problems encountered in trying to obtain an analog computer solution are delineated.

The writer wishes to express his appreciation for the assistance given him by Professor John E. Brock and to the Professors, particularly Professor O. H. Polk, and the technicians of the Electrical Engineering Department. The numerical solutions in Appendix I were contributed by Professor Brock.

2. Background.

Solutions for many dynamics problems have been established using the analog computer and references can be found in the technical literature. One such reference for a non-linear problem, Analog Computer Solution of a Non-Linear Differential Equation, by H. G. Markey, (2)[2], was found but was only applicable in a general way to this investigation.

It was considered that if a means could be found to display some of the classical problems encountered in early college dynamics on the

[1] d/dt is denoted by a dot placed above the variable operated on.
[2] Numbers in parentheses refer to references in Bibliography.

analog computer the following advantages would be obtained:

(a) the general usefulness of the analog computer could be made more readily apparent;

(b) in dealing with these problems attention could be focused on the dynamic principles leading to the governing differential equations and upon the mechanical significance of the results and not upon the mathematical difficulties in obtaining an analytical solution;

(c) in the case of those problems where analytical solutions have been obtained for certain particular parameter values and which therefore seem to be separated into many different cases the dynamical significance of which is not apparent, the general problem could be dealt with directly;

(d) it would be possible to include normal dynamical influences (such as energy loss due to pivot friction or windage) without so complicating the mathematics of the solution that the modified problem appears to be entirely different from the idealized problem.

In addition to the above it was desired to obtain these results using only the analog computers and their associated equipment normally available in an analog computer laboratory.

3. The problem and general method of solution.

The problem considered was that of determining the subsequent motion of a body weighing 1930 pounds, attached to a spring having a free length of five inches and a scale of ten pounds per inch, when released with the following initial conditions. At the initial instant the radius is four inches and its rate of change is zero; the polar angle, θ , is zero and its rate of change is three radians per second.

The other end of the spring is attached to a fixed point and the body is permitted to slide without friction upon a horizontal plane.

We will discuss the sequence of steps necessary for the solution of a problem of this type, and then we will proceed with the solution. One might of course proceed with a full scale experimental program as a method of solution, but eliminating this possibility we would:

a. Using the principals of Mechanics arrive at one or more differential equations describing the motion.

b. Solve these equations, incorporating the starting conditions. This solution may be analytic, numerical, experimental (dealing with, possibly, scaled down mechanical variables), or by means of an analog, in which one deals experimentally with variables of another type (such as electrical) which satisfy similar differential equations.

c. Interpret the mathematical, experimental, or analog results in the proper mechanical light so as to arrive at a meaningful solution to the original problem.

In this thesis, we are investigating the practicability of proceeding immediately from the first step to a solution by use of standard analog computer equipment. We do not have an analytical solution to the problem stated, although it is likely that one might be obtained in terms of elliptic functions and integrals. However, in an appendix we will develop two different numerical solutions to the problem with which we can compare our analog solutions.

4. The differential equation of motion.

Fig. 1 shows the body in a general position. The solid arrow represents the spring force $F_s = 10(r-5)$, where r represents the radial distance from the fixed point 0. The dotted arrows represent D'Alembert forces necessary for dynamic equilibrium. We see that

$$F_s + ma_r = 0$$
$$ma_\theta = 0$$

Now by kinematics, $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$. From the second equation we see that $r^2\dot{\theta} = C = \text{Const.}$ This can also be seen from the fact that the angular momentum of the system about 0, namely $mr^2\dot{\theta}$, is invariable. From the first equation, we have $10(r-5) + \frac{5 \times 384}{384}(\ddot{r} - r\dot{\theta}^2) = 0$, and from this we get $2r - 10 + \ddot{r} - r\dot{\theta}^2 = 0$. Substituting $\dot{\theta} = C/r^2$ we finally get $\ddot{r} = \frac{C^2}{r^3} - 2r + 10$. In our case, evaluating C at the initial instant we have C = 48. Thus we have as our set of differential equations:

$$\ddot{r} = \frac{2304}{r^3} - 2r + 10$$
$$\dot{\theta} = 48/r^2$$

Now it is possible to perform some mathematical manipulations which simplify this system. In particular, it is easily possible to obtain a first integral of the first equation of the system. However, we regard it as contrary to the spirit of this thesis, the purpose of which deals with the ready feasibility of making an analog computer solution of this system, to perform any such manipulations, and it is

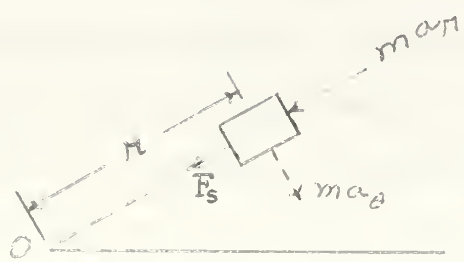


Fig. 1 Force Diagram for Central Force Problem

this system with which we shall be directly concerned when we attempt the analog solution. The numerical solutions for this problem will be found in Appendix I.

5. Discussion of equipment.

Before taking up the solution of the problem, a description of the equipment used in the solution of this problem will be presented. It is assumed that the reader is already acquainted with the basic theory of the analog computer and with the usual circuitry used, such as summers, integrators, etc. The Handbook of Automation, Compution and Control, Vol. 2, E. M. Grabbe, (1), is a good reference for this information as well as for additional information on the equipment discussed below.

A. Donner Analog Computer, Model 3000.

This analog computer, which can be used for the quantitative solution of linear (and certain classes of non-linear) differential equations and transfer functions, contains ten DC operational amplifiers, any one of which can be used for addition, subtraction, multiplication or division by a constant, sign changing, or integration. Problems expressed as differential equations are entered in terms of electrical components on a detachable problem board. Stability and accuracy are satisfactory for problem solution times up to 100 seconds or more which permits accurate recording with conventional pen recorders. (5)

B. Donner Function Multiplier, Model 3730.

This function multiplier consists of two multiplier channels and a regulated power supply. Each multiplier channel produces an output voltage which is accurately proportional to the instantaneous produce of two independent input voltages, where each input is either

constant or varying with time. Either input may be positive or negative, so that four quadrant multiplication is provided. The range of output and input voltages is plus and minus 100 volts; this being necessary to stay within the linear range of the operational amplifiers of the computer. To maintain the output voltage at 100 volts or less the Function Multiplier is designed to give an output voltage which is .01 the product of the input voltages. (6)

For the solution of the problem of this thesis two of these multipliers were used. They gave accurate results when used for straight multiplication although they do tend to drift over a period of time and have to be readjusted; this is a minor operation, however. When used in a division circuit, which is discussed in a later section, the results obtained were not as accurate, however. It is believed, however, that this was a fault of the circuit and not of the function multiplier bacause, as mentioned above, the function multipliers gave quite accurate outputs when used for multiplication alone.

C. Donner Function Generator, Model 3750.

This variable base function generator is designed for use in conjunction with two external operational amplifiers to produce an output voltage which can be adjusted to approximate a desired single valued function of the input voltage. One operational amplifier is required for operation of the function generator and the other is used to accept the output signal at its summing junction. This amplifier may also be used for additional summing, inverting, integrating or other operations. The function generator operates on the principal that the function can be

approximated by a series of straight line segments, each segment being limited to a slope between plus and minus two volts per volt. The input and output voltages may vary between plus and minus 100 volts. (4)

For the solution of the thesis problem it was desired to use this function generator to generate the function $2304/r^3$ but it was found that the slope of curve for this function exceeded the capability of the equipment. This is duscussed further in Sec. 9. If it had been possible to use this function generator the two function multipliers would not have been required.

6. Computer equation and scaling.

To reduce our problem to a form suitable for the computer it is necessary to apply scaling factors. This was done using the methods outlined in Basic Theory of the Electronic Analog Computer, by R. C. H. Wheeler, (9). A brief summary of this process is presented here.

The differential equation to be solved is first arranged so as to solve for the derivative of the highest order. In our case $\ddot{r} = \frac{A}{r^3} - BX + C$. The equation is then scaled so that maximum value of each parameter is represented by a voltage, close to but not exceeding 100 volts. To do this scaling factors are assigned as shown by the following example:

$$x = \alpha_x X$$

Here X is the computer voltage representing the variable x, and α_x is its scaling factor. After being calculated the scaling factor is usually rounded off to facilitate computations. After suitable scaling factors are found, the equation is put into the form:

$$\alpha_{\ddot{r}} \ddot{R} = \frac{A}{(\alpha_r R)^3} - B\alpha_r R + C$$

$$\ddot{R} = \frac{A}{\alpha_{\ddot{r}}\, \alpha_r^3 R^3} - \frac{B\alpha_r R}{\alpha_{\ddot{r}}} + \frac{C}{\alpha_{\ddot{r}}}$$

To develop the applicable circuits for the problem solution it is necessary to determine the values of resistance and capacity needed for each component of the circuit. Using the procedures in Wheeler's book, (9), pages 2-10 we find, for example, that an operational amplifier when used as a summer has an output voltage $e_0 = -(\omega_1 e_1 + \omega_2 e_2 + \quad)$ or in our case $\ddot{R} = -(\omega_1 \frac{A}{R^3} - \omega_2 R + \omega_3 C)$. If we now let $\omega_i = \frac{a_i R_f}{R_i}$ where a_i is a coefficient potentiometer setting and R_f and R_i are resistances, we can establish the relationship $a_i = \frac{\omega_i R_i}{R_f}$. It should be noted here that an R with a subscript, R_i, refers to a resistor and, R, without the subscript refers to the voltage representing the variable r, the radius of the problem. Now the above relationship can be solved for a_i. For integrators the relationship is $\omega_i = \frac{a_i}{R_i C_f}$, where C_f refers to a capacitor.

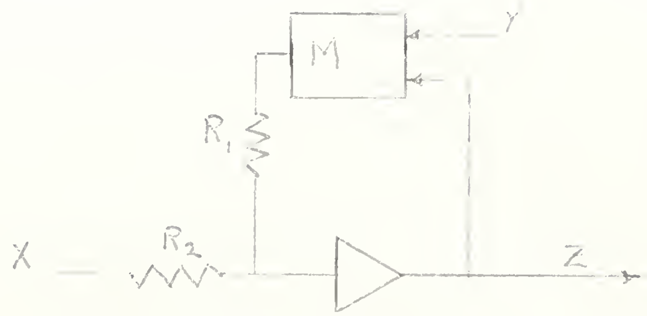
7. Analog computer circuits.

In Appendix II the calculations for scaling the differential equations of our problem are presented. After scaling we have the following equations:

$$\ddot{R} = Z - .8R + 20 \qquad Z = \frac{576,000}{R^3}$$

$$\dot{\theta} = 30,000/R^2$$

Before solving this problem on the analog computer two main decisions have to be made: first how to calculate R^3 and R^2, and second how to develop the terms $Z = 576,000/R^3$ and $\dot{\theta} = 30,000/R^2$. It was hoped at first that the terms for Z and $\dot{\theta}$ could be developed using function generators but as mentioned previously this proved unsatisfactory. Thus it was expedient to use the division circuit shown in Fig. 2.



Division Circuit (6)

Fig. 2

With this circuit $Z = \frac{100R_1 X}{R_2 Y}$. The factor 100 results from the output of the multiplier being .01 YZ. If we now let $Y = R^3$ and $X =$ constant, using the above relationship we should be able to develop $Z = 576{,}000/R^3$.

We know from the parameters of the problem that when r is 4, R should be 20 volts. If we then put this value through two function multipliers we come up with K^2R^3. As this value is small, .8 volt, we multiply it by a factor of two using an operational amplifier and then put it into the function multiplier of the division circuit. Also using this value of the voltage for R we can calculate the value Z should have, in this case 72 volts. With these values we can now solve for a value of X so that with an input of 1.6 volts for Y and the calculated value of X , Z will be 72 volts. Solving for X:

$$X = \frac{ZYR_2}{100R_1} = \frac{1.6 \times 72\, R_2}{100\, R_1} = \frac{115 R_2}{100 R_1}$$

Now letting R_2 equal 10M and R_1 equal 1M, we find that an X of 11.5 should be used. (It was found that resistances of 10M and 1M worked better than resistors of 0.1M and 1M). This same procedure was applied to $\dot{\theta}$ and the corresponding voltage, X_1 was found to be 30 volts.

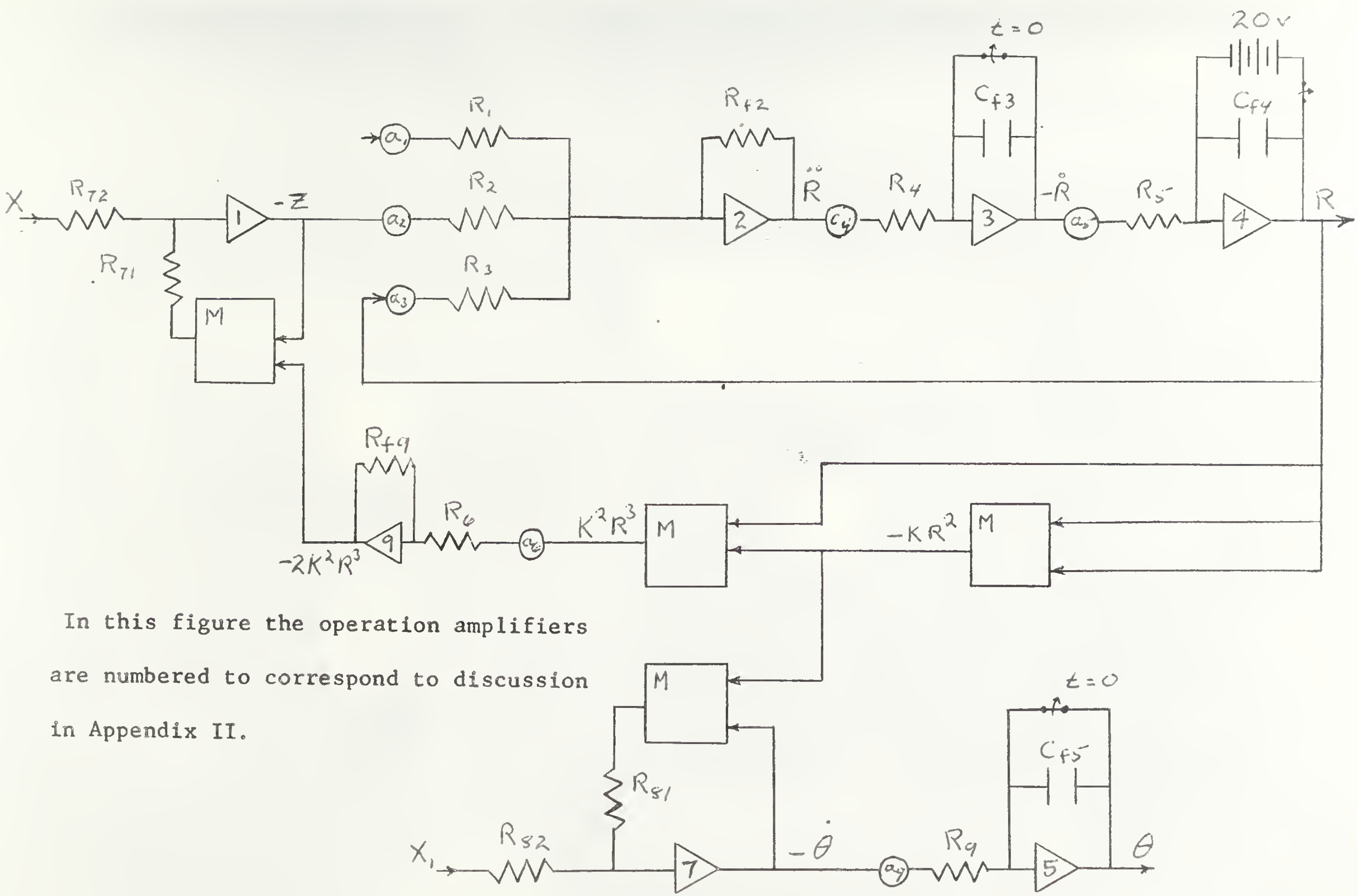
It should be noted here that another method for determining $\dot{\theta}$ presents itself, that of multiplying Z by R/19.2. By doing this the second division circuit could be eliminated and only another multiplication, with its more accurate results, required. This method was tried and it was found that for some unexplained reason $\dot{\theta}$ passed through zero and became slightly negative. As a result of this θ oscillated instead of increasing smoothly from zero to a maximum value. For this reason the division circuit for developing $\dot{\theta}$ was used.

After the above determinations were made, the circuit of Fig. 3 was assembled and computations made. In assembling the circuit the values of the a's calculated in Appendix II were adjusted for the actual values of resistances and capacitors used, e.g., 1.005 M actual resistance vs. nominal value of 1M. Figs. 4 and 5 are photographs of the setup used and shows the relative simplicity of the final setup for solution of the problem.

8. Results.

After assembling the circuit of Fig. 3, it was found that to obtain the desired values of voltage for Z and $\dot{\theta}$ the values of the input voltages calculated for X and X_1 had to be adjusted. For X, a value of 20 volts, and for X_1 a value of 33 volts was required. Once these adjustments were made, the computing runs were made and the results are shown on the Brush Recordings of Fig. 6 and Fig. 7. These recordings were all made using a paper speed of 5/mm/sec and with varying voltage scales as shown on each trace. These results are also summarized in the table of Fig. 8. From these results curves were plotted and then compared with the results obtained by the numerical solutions, as shown in Figs. 9, 10, & 11.

In analysing the results each term will be considered separately. Considering $\ddot{r}$ first it is seen that the maximum value of 38 obtained agrees with the maximum value of the numerical solution but that the minimum value of - 12.5 is lower than the - 15.26 of the numerical solution. This latter discrepancy is attributed to the actual values obtained for Z and will be discussed later.



In this figure the operation amplifiers are numbered to correspond to discussion in Appendix II.

Fig. 3 Circuit Diagram

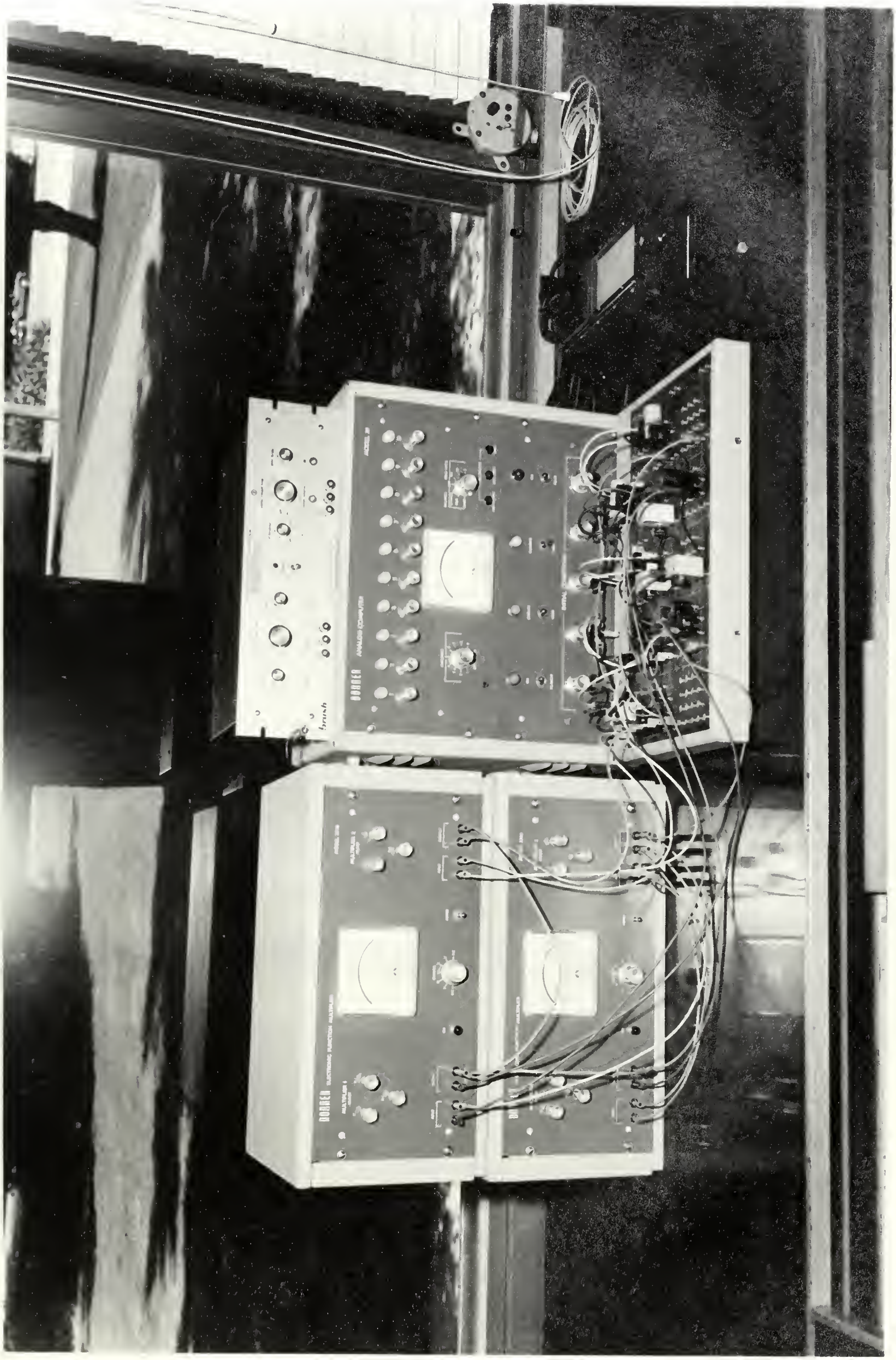


Fig 5

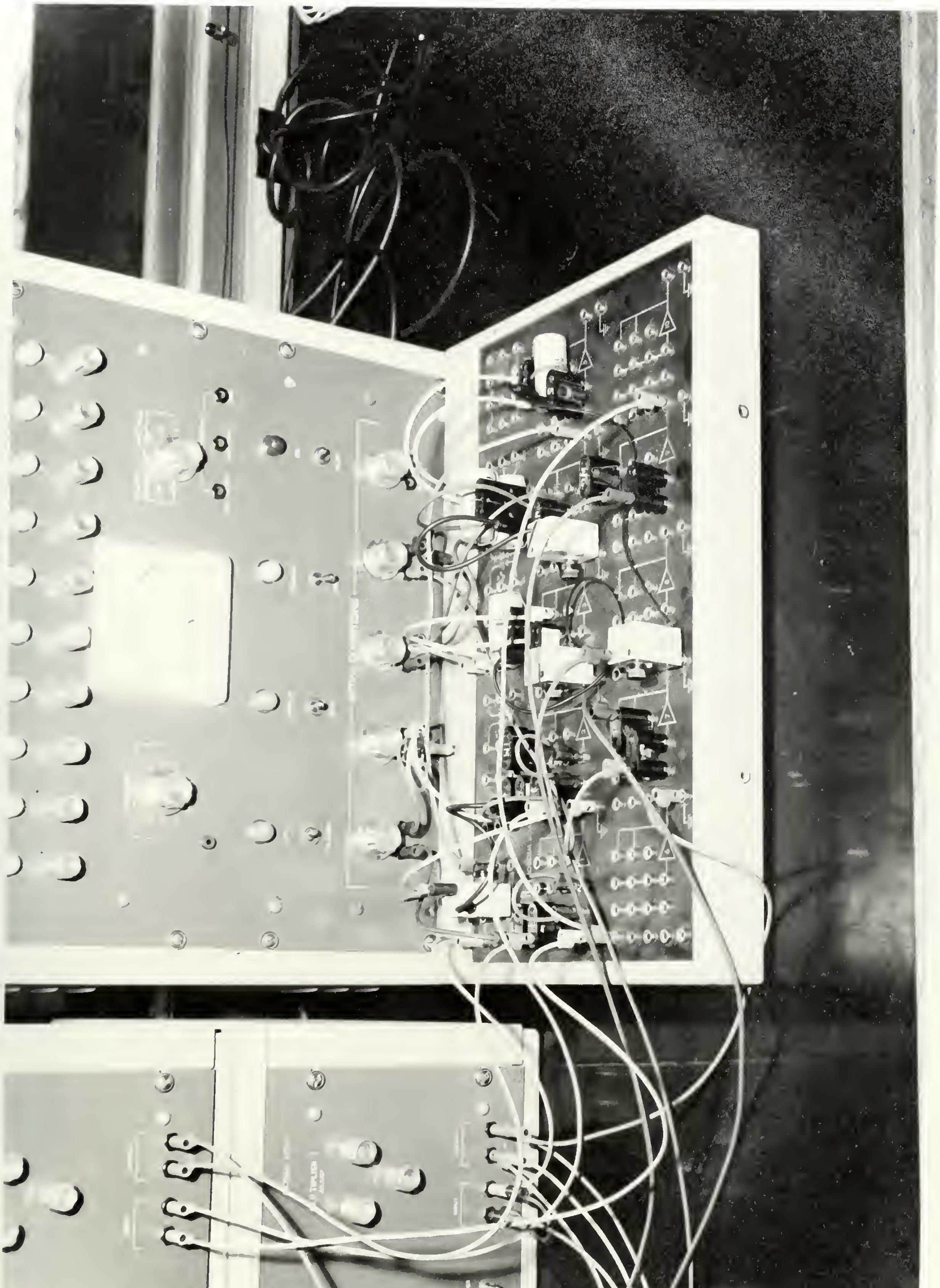
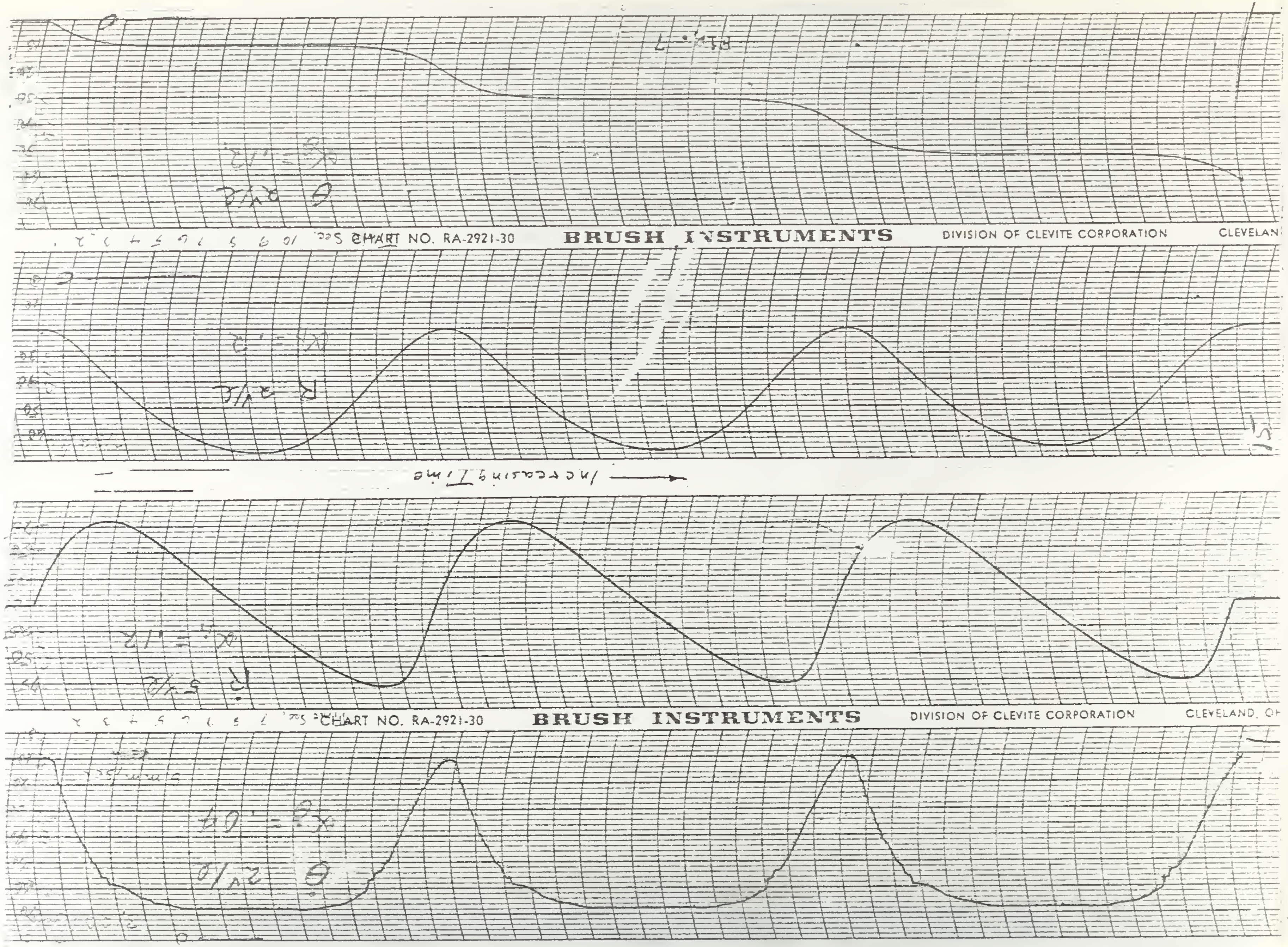


Fig. 5



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BRUSH INSTRUMENTS
DIVISION OF CLEVITE CORPORATION
CLEVELAND, OH

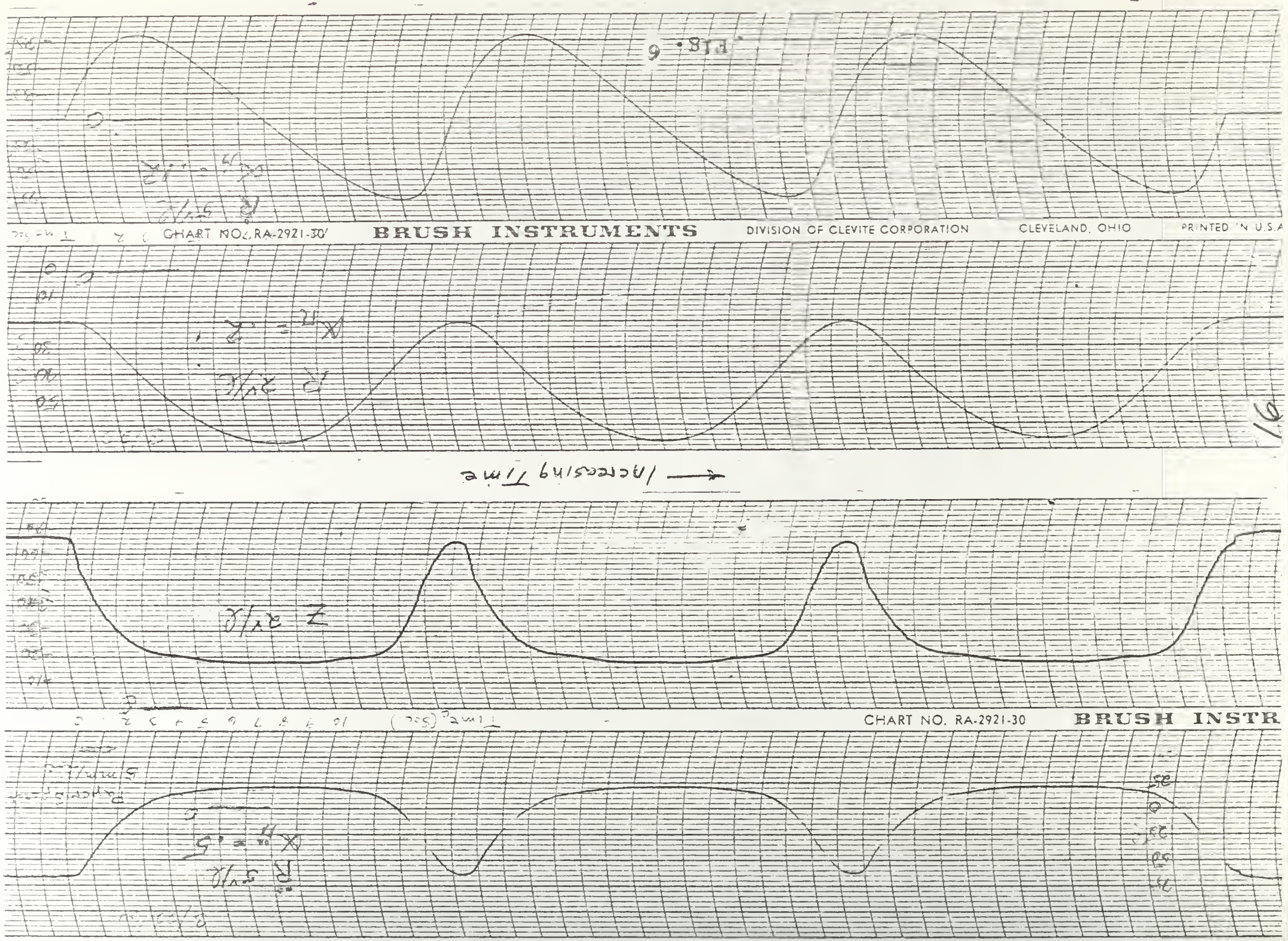
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Fig. 6

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Summary of Analog Results												
t_c	t_p	Z	$\ddot{R}$	$\ddot{r}$	$\dot{R}$	$\dot{r}$	$\dot{\Theta}$	$\dot{\theta}$	R	r	Θ	θ
0	0	76	76	38	0	0	74	2.96	20	4	0	0
	.1	62	69	34.5	35	4.2	68	2.72			3	20.6
1	.2	49	47.5	23.7	57.5	6.9	52	2.08	24	4.8	5.5	37.8
	.3	38	27.5	13.7	71.5	8.6	41	1.64			7	48.1
2	.4	30	12.5	6.7	80	9.6	31	1.24	34	6.8	8.5	58.3
	.5	26	2	1	81	9.7	23	.92			9.5	65.5
3	.6	23	- 7	- 3.5	80	9.6	20	.80	44	8.8	10	68.7
	.7	22	-12	- 6	75	9.0	18	.72			11	75.5
4	.8	21	-15	- 7.5	67.5	8.1	15	.60	53	10.6	11	75.5
	.9	20	-17	- 8.5	60	7.2	13	.52			11	75.5
5	1.0	19	-20	-10	50	6.0	12	.48	60	12.0	11	75.5
	1.1	18	-21	-10.5	42.5	5.1	11	.44			11	75.5
6	1.2	18	-22	-11	32.5	3.9	11	.44	64	12.8	11	75.5
	1.3	18	-23	-11.5	22.5	2.7	11	.44			11	75.5
7.0	1.4	18	-24	-12	12.5	1.5	11	.44	66	13.2	11	75.5
7.4	1.48	18	-25	-12.5	0	0	11	.44	67	13.4	11	75.5
[illegible]		$\alpha_R = .5$		$\alpha_R = .12$		$\alpha_\theta = .04$		$\alpha_R = .2$		[illegible]		

Fig. 8

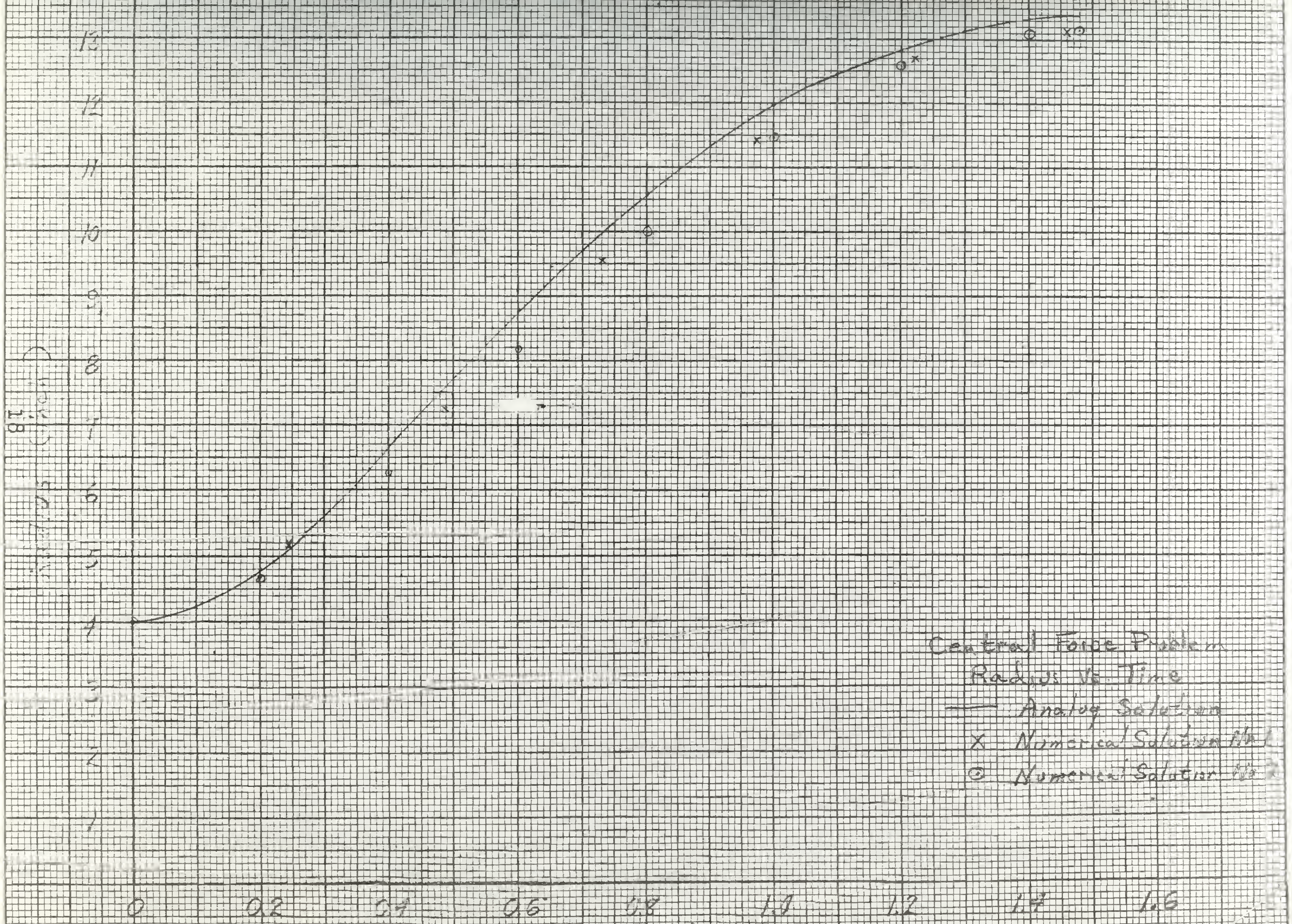
Central Force Problem
Radius vs Time
Analog Solution
X Numerical Solution
⊙ Numerical Solution
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1
2
3
4
5
6
7
8
9
10
11
12
13

Fig. 9

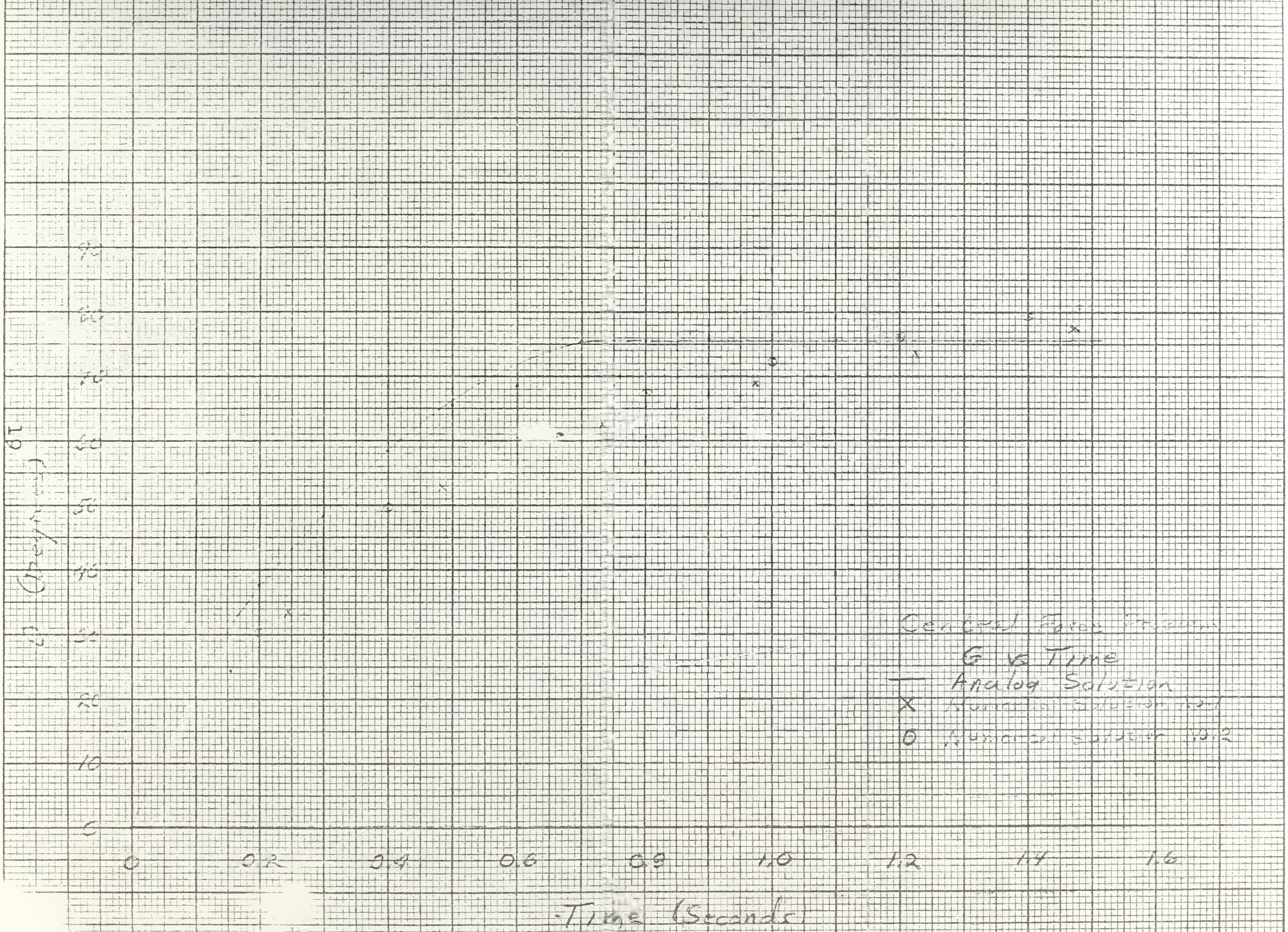

Fig.10

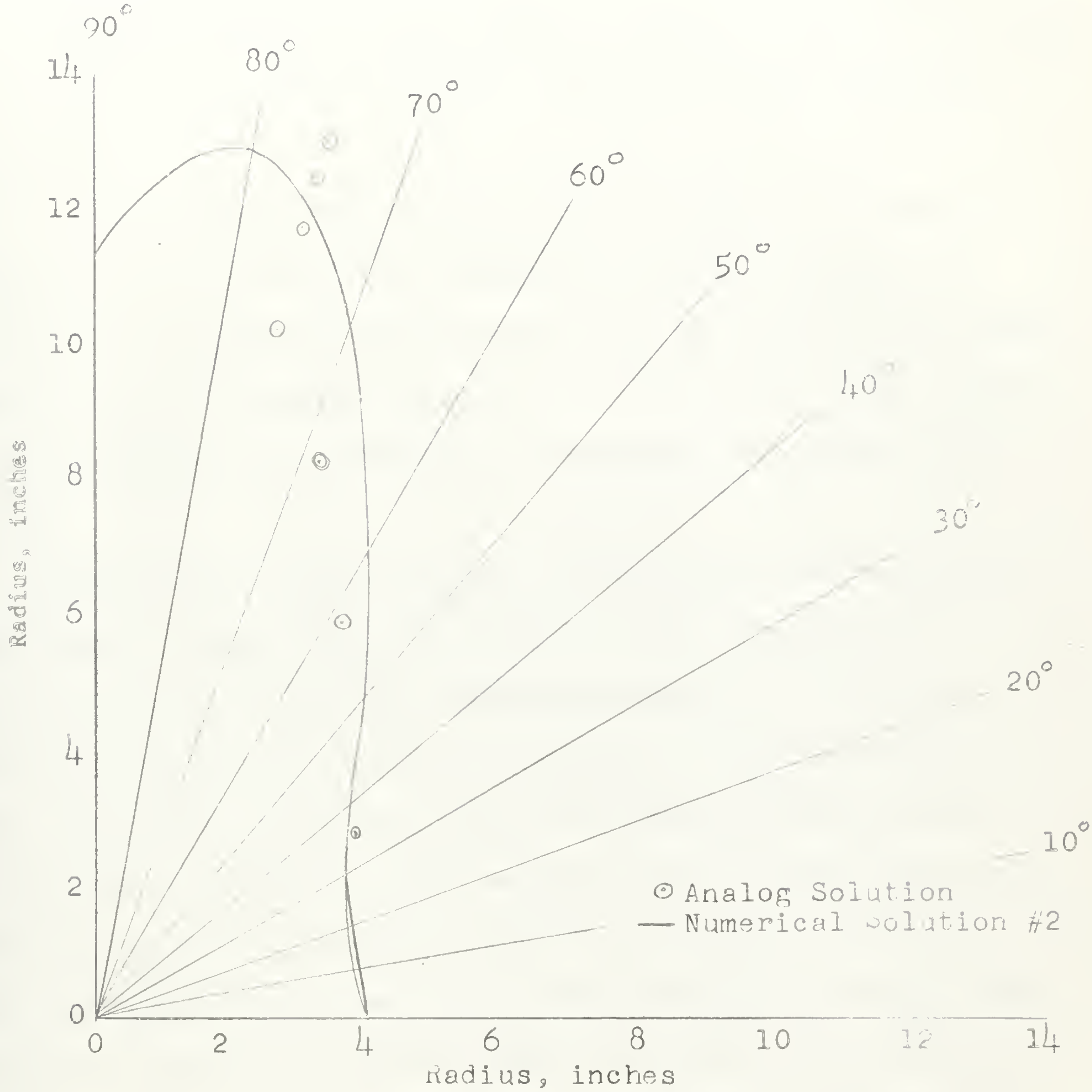

Fig 11, Radius vs Angle of Rotation

For $\dot{r}$ the maximum value obtained was 9.7 and the minimum values were zero when r was a maximum and a minimum. This agrees well with the numerical solution where the maximum value was 9.6. Considering r we see from Fig. 9 that the analog values are slightly higher at all values than the r's of the numerical solution. This error is not considered excessive.

The largest discrepancies appear when we consider θ. As can be seen in Figs. 8, 10, and 11 the analog value reached its maximum for the first apse (point of greatest distance from the center of attraction) rapidly and then remained constant for a period of time. Here as with $\ddot{r}$ the discrepancies are considered to be caused by the values obtained for $\dot{\theta}$.

Considering the problem overall, the more significant results obtained appear to agree rather well with the values obtained by the numerical solutions. The major discrepancies appear when the part of the circuit handling the division is considered. As can be seen from Figs. 6 and 12 for Z, and Fig. 7 for $\dot{\theta}$, the outputs of these division circuits change rapidly to a small negative voltage and then remain relatively constant for a period of time. We can also see from Fig. 10 that the division circuit does not do what theoretically it should. Thus for either parameter the minimum voltage desired, when r is a maximum is never obtained. With Z, this term is small when compared with the others in summing for $\ddot{r}$ and the effect is not pronounced. With $\dot{\theta}$ however this defect has a more pronounced effect and θ is not developed in the smooth curve desired.

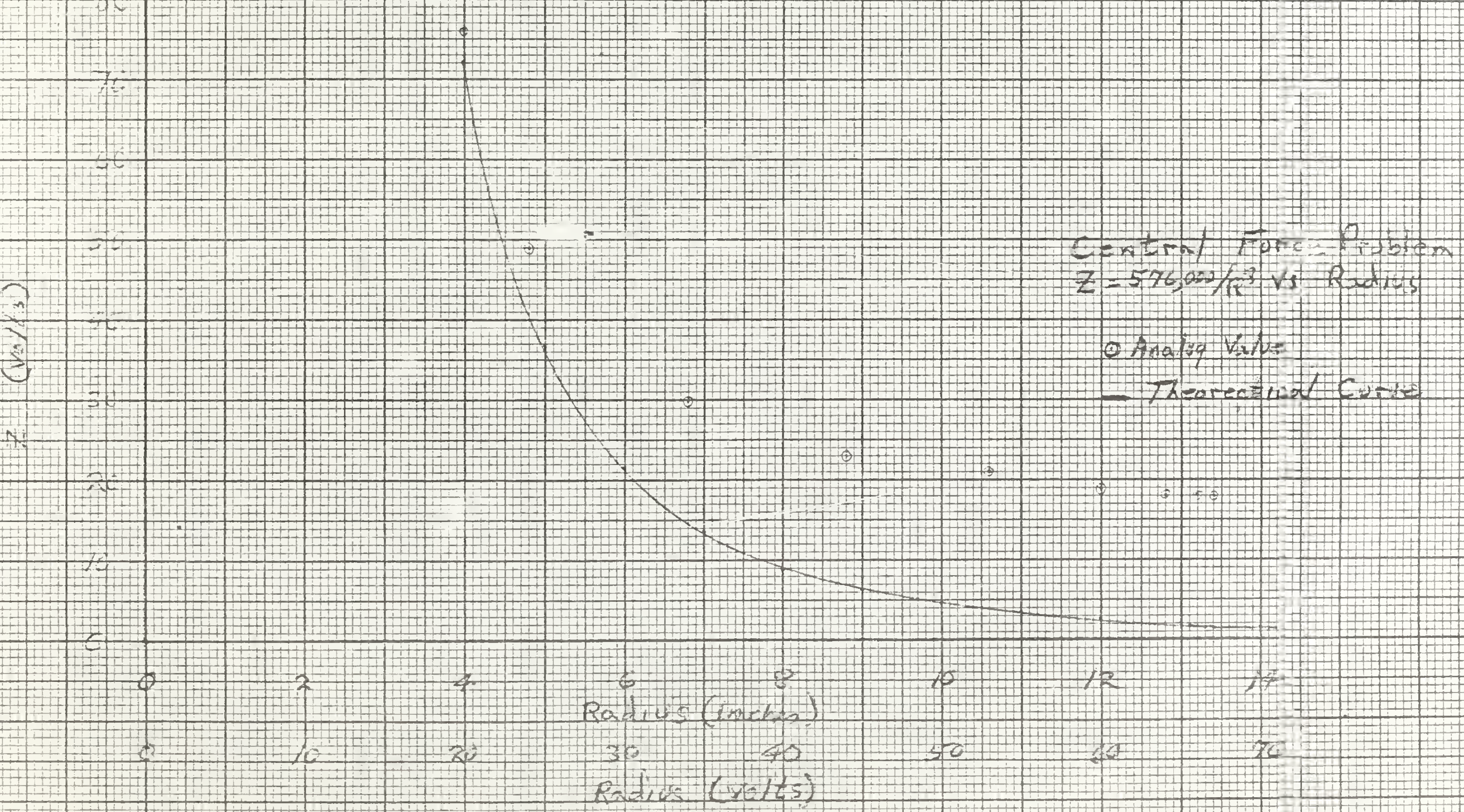
Central Force Problem
Z = 576,000/R³ Vs Radius
⊙ Analog Value
— Theoretical Curve
0
2
4
6
8
10
12
14
Radius (inches)
0
10
20
30
40
50
60
70
Radius (volts)

Fig. 12

9. Discussion of discrepancies.

The discrepancies found in the above problem solution were attributed to the division circuits used. No satisfactory answer could be found as to why the desired divisions could not be obtained. It is known that a circuit such as this develops a certain amount of noise. That is, the function multiplier has a certain amount of noise inherent in its output and that if this is put through an operational amplifier this noise is amplified. The Handbook of Automation, Computation and Control, Vol. 2, by Grabbe (1) discusses this briefly and mentions that a small capacitor placed in parallel with the multiplier will help to alleviate this problem. This was tried but did not give satisfactory results.

As mentioned previously, if a function generator could have been used, the circuitry could have been simplified, i.e., no function multipliers would have been required. With the Donner function generator the slope of the function $576{,}000/R^3$, for low values of R, exceed the maximum of two volts per volt permitted by the device. One other type of function generator was tried. This was an Autograff XY plotter converted to a function generator by replacing the recording pen with a pick-up coil and plotting the desired function with a conducting ink. However, with this arrangement the desired range of voltages could not be obtained.

Still another type of function generator that might have proved satisfactory, if it had been available, is the photo-former type. This type of function generator operates as follows. The basic piece of equipment is a cathode-ray tube. An input voltage is applied between the

horizontal deflection plates of a cathode ray tube through a suitable d-c amplifier. The voltage between the vertical deflection plates is taken as the output voltage. This voltage is made to vary as a function of the input voltage by a feed-back arrangement which forces the electron beam to follow the boundary of an opaque mask placed over the lower portion of the cathode-ray screen. Thus as the spot on the cathode-ray tube screen emerges from behind the mask a photocell in front of the tube applies an error voltage across the input terminals of the vertical deflection d-c amplifier, so phased that the beam is forced downward toward the face of the mask. Therefore if the mask is shaped to represent the function being generated the spot will follow this curve and deliver an output voltage proportional to the input voltage. This type of function generator is said to be very accurate in developing many functions. (3)

10. Conclusions.

Considering the results obtained from this problem (keeping in mind that indeed it is but a single problem), it was found that a "typical" non-linear dynamics problem can be set up on an analog computer. However this type of set-up is not done rapidly or easily. Considerable thought has to be given as to what type of equipment shall be used and what kind of circuits are necessary. Because they require the use of various types of non-linear computer accessories the circuits become very sensitive and results accurate to the degree normally expected from the analog computer may not be obtained. Care has to be taken in selecting scaling factors, where powers and roots are involved, to avoid over-loading the operational amplifiers. It was found, however, that the function multipliers used

to square and cube R gave quite accurate results, even at low volt-ages, providing they were kept balanced.

In setting up a problem of this type it will usually be found that there will be one key term to be developed, such as the A/R^3 of this problem. Once a way is found to develop or represent this term the remaining computer setup is routine and with patience and luck a solution can be obtained.

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APPENDIX I

Numerical Solutions for Central Force Problem

The statement of the problem is given in Sec. 3 on page 2. Representing this information in mathematical terms, we have F_{spring} = 10(r-5) lbs. and m = 1930/386 = 5 lbs sec^2/in., and initially (at time = 0) we have r = 4 inches, $\dot{r}$ = 0, θ = 0, and $\dot{\theta}$ = 3 radians/sec. Since energy is conserved, we have

$$E = T + V = \frac{1}{2}(r^2\dot{\theta}^2 + \dot{r}^2) + (r-5)^2 = \text{Constant}$$

where E, T, and V are expressed as energy per unit mass in units of in^2/sec^2. Here we have used $V = [\frac{1}{2}k(r-5)^2] \div m = (r-5)^2$

Using the initial conditions to evaluate E, we have

$$E = \frac{1}{2}(4^2 \cdot 3^2 + 0) + (4-5)^2$$
$$= 72 + 1$$
$$= 73$$

At apses $\dot{r} = 0$ $\therefore$ Apsidal radii are given by $\frac{1}{2} r^2 \dot{\theta}^2 + (r-5)^2 = 73$ Substituting $\dot{\theta} = 48/r^2$ and rationalizing we get:

$$\frac{1}{2} r^2 \frac{48^2}{r^4} + (r-5)^2 = 73$$
$$1152 + r^4 - 10r^3 + 25r^2 = 73r^2$$
$$r^4 - 10r^3 - 48r^2 + 1152 = 0$$

Knowing that one root is 4 we obtain $(r-4)(r^3 - 6r^2 - 72r - 288)$

This can have only one positive root. Synthetic division indicates a root of approximately 13.2 and using Newton's method:

$$f(r) = r^3 - 6r^2 - 72r - 288$$
$$f'(r) = 3r^2 - 12r - 72$$
$$a_1 = 13 - \frac{f(13)}{f'(13)} = 13 - \frac{(-41)}{279} = 13.15$$
$$a_2 = 13.15 - \frac{f(13.15)}{f'(13.15)} = 13.15 - \frac{1.596}{288.97} = 13.1445$$
$$a_3 = 13.1445 - \frac{f(13.1445)}{f'(13.1445)} = 13.1445 - \frac{.00756}{288.5996}$$
$$= 13.144738$$

To find the apsidal angle and r and θ as functions of time, we resort to a numerical procedure since the integral involved is not elementary. Returning to fundamentals we have:

$$T + V = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + (r-5)^2 = 73$$
$$h = r^2\dot{\theta} = 48$$
$$\ddot{r} - r\dot{\theta}^2 = -2(r-5)$$
$$\therefore \ddot{r} = \frac{2304}{r^3} - 2r + 10$$

We also know $r_1 = 4$ and $r_2 = 13.144$. Now using an iterated integration, a curve of $r = r(t)$ is assumed such that $\dot{r} = 0$ at the end points (apses). The apsidal time τ is divided into n equal intervals τ/n ; τ being as yet unknown. We will use $n = 6$, although a larger n will give a more accurate result.

Assumed values of r are selected for each epoch. Values of $\ddot{r}$ are calculated and integrated with the condition $\dot{r}=0$ at $t=0$. This should yield $\dot{r}=0$ at $t=\tau$, but there is an error ϵ . We remove this error by using a correction curve which is essentially $\Delta\dot{r} = \frac{(2t^3 - 3t^2\tau)\epsilon}{\tau}$ expressed however in appropriate form for and obtained by numerical integration. This arises from assuming that the error in $\dot{r}$ is due to an error in $\ddot{r}$ which must be essentially parabolic in nature, vanish-

ing at the end points since the apsidal distances are known. The rest of the calculations are self explanatory and lead to the curved path shown in Fig. 9.

t	0		1		2		3		4		5		6	τ/6
n	4.000		5.100		7.310		9.610		11.520		12.750		13.144	
n^3	64.00		133.43		390.62		887.50		1528.82		2072.67		2271.07	
$2304/n^3$	36.000		17.267		5.898		2.596		1.517		1.112		1.014	
$\ddot{n}$	32.000		17.047		1.298		-6.624		-11.513		-14.388		-15.274	
$\dot{n}$	0	19.000		36.047		37.325		30.701		19.188		4.800	-2.837	
$\Delta \dot{n}$	0	0.000		.405		1.054		1.783		2.432		2.837	-2.837	
$\dot{n}$	0	19.000		36.452		38.379		32.484		21.620		7.637	0	τ/6
$n-4$	0		19.000		55.452		93.831		126.315		147.935		155.572	[illegible]
$n-4$	0		1.117		3.259		5.515		7.425		8.696		9.141	in
n	4.000		5.117		7.259		9.515		11.425		12.696		13.144	in
Repeat Twice														
n	4.000		5.120		7.256		9.509		11.420		12.693		13.144	
n^2	16.000		26.214		56.650		[illegible]		130.416		161.112		[illegible]	
$\dot{\theta}$	3.000		1.831		.912		.531		.368		.298		.278	[illegible]
θ	0	1.500		3.331		4.243		4.774		5.142		5.440	5.579	τ/6
θ	0	20.87		46.34		59.03		66.42		71.54		75.68	77.62	Degrees
θ	0		32.60		52.68		62.72		68.98		73.61		77.62	Degrees

$$\frac{\tau^2}{36} = \frac{9.144738}{155.01} = .0589582$$

$$\frac{\tau}{6} = .242813$$

$$\tau = 1.457 \text{ sec}$$

Fig. A13, Table of Numerical Integration

Second Numerical Method

From the expression for E, $r^2\dot{\theta}^2 + \dot{r}^2 + 2(r$ [illegible] .
Upon substituting $\dot{\theta} = 48/r^2$, we get $\dot{r}^2$ as a function of r :
and thus can construct a curve of $\dot{r}$ as a function of r (We take the positive branch of the square root so as to deal with the period during which r is increasing from 4 to 13.145 inches.). Also we have

$$\ddot{r} = \frac{2304}{r^3} - 2r + 10$$

so that we can construct a curve of $\ddot{r}$ as a function of r, and this relation shows that $\ddot{r} = 0$ when r is approximately equal to 7.62. Having curves of both $\dot{r}$ and $\ddot{r}$ as functions of r, we can construct a curve of $\ddot{r}$ as a function of $\dot{r}$.

The differential of time may be written in either of two ways

$$dt = \frac{dr}{\dot{r}} \text{ or } \frac{d\dot{r}}{\ddot{r}}$$

, and this permits us to write

$$\tau = \int_{\dot{r}(4)}^{\dot{r}(6)} \frac{d\dot{r}}{\ddot{r}} + \int_{6}^{9} \frac{dr}{\dot{r}} + \int_{\dot{r}(9)}^{\dot{r}(13.145)} \frac{d\dot{r}}{\ddot{r}}$$

so as to avoid infinite values for the integrands. These calculations can be carried out by numerical methods as shown on the following pages and τ is found to be 1.4863 seconds, which agrees with the value of τ calculated in the first numerical solution.

r	r^3	$2304/r^3$	$\ddot{r}$	$2304/r^2$		$\dot{r}^2$	$\dot{r}$	$1/\dot{r}$	$1/\ddot{r}$
4	64	36.0000	38.000	144.000		0	0		.026316
4.5	91.125	25.2840	26.2840	113.7780	.2500	31.7220	5.6322		.038043
5	125	18.4320	18.4320	92.1600	0	53.8400	7.3376		.054253
5.5	166.375	13.8482	12.8482	76.1651	.2500	69.3349	8.3268		[illegible]
6	216	10.6667	8.6667	64.0000	1.0000	80.0000	8.9443	.11180	.115385
7	343	6.7172	2.7172	47.0204	4.0000	90.9796	9.5383	.10484	
8	512	4.5000	-1.5000	36.0000	9.0000	92.0000	9.5917	.10426	
9	729	3.1605	-4.8395	28.4445	16.0000	95.5556	9.2496	.10811	-.206633
10	1000	2.3040	-7.6960	23.0400	25.0000	72.9600	8.5417		-.129938
11	1331	1.7310	-10.2690	19.0410	36.0000	54.9590	7.4134		-.097380
12	1728	1.3333	-12.6667	16.0000	49.0000	32.0000	5.6569		-.078947
13	2197	1.0487	-14.9513	13.6331	64.0000	[illegible]	2.0897		-.066884
13.144	2271.065	1.0145	-15.2744	13.3351	66.3325	0	0		[illegible]

Fig. A14 Table of Calculated Values

12ND 6961 NAVY-DPPO-12ND-MI

t	$\dot{r}$	r	$\dot{\theta}$	t	r	$\theta_{Rad.}$	θ°
0	0	4.00	3.00				
.0545	2	4.10	2.855	0	4.00	0	0
[illegible]	4	4.28	2.620	.2	4.66	.529	30.31
.1859	6	4.58	2.288	.4	6.24	.867	49.68
.2895	8	5.30	1.709	.6	8.17	1.058	63.62
[illegible]	8.9443	6.00	1.333	.8	10.00	1.175	67.32
[illegible]		6.50	1.136	1.0	11.47	1.259	72.14
[illegible]		7.00	.980	1.2	12.56	1.326	75.97
[illegible]		7.50	.852	1.4	13.10	1.384	79.30
.5841		8.00	.750	1.4863	13.14	1.409	80.73
[illegible]		8.50	[illegible]				
.6899	9.2496	9.00	[illegible]				
[illegible]	9	9.44	.539				
[illegible]	8	10.55	.431				
1.0644	6	11.84	.342				
1.2159	4	12.52	.306				
[illegible]	2	13.01	.284				
1.4863	0	13.144	.278				

Fig. A15, Table of Calculated Values

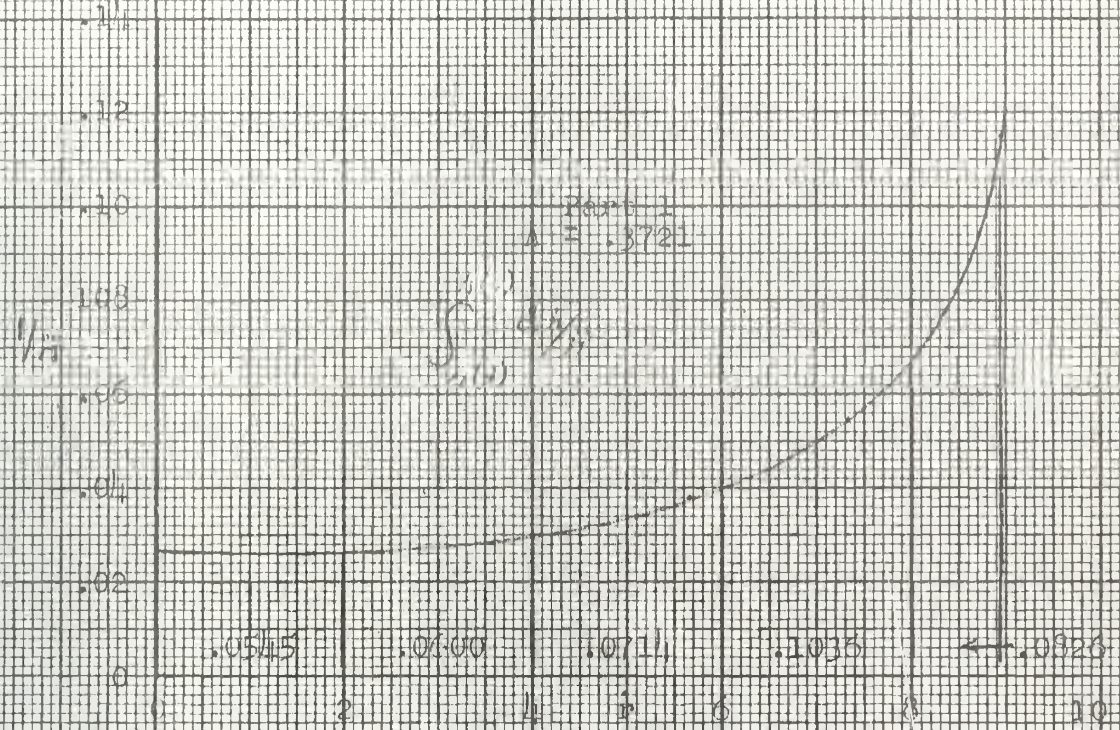

Fig. A16

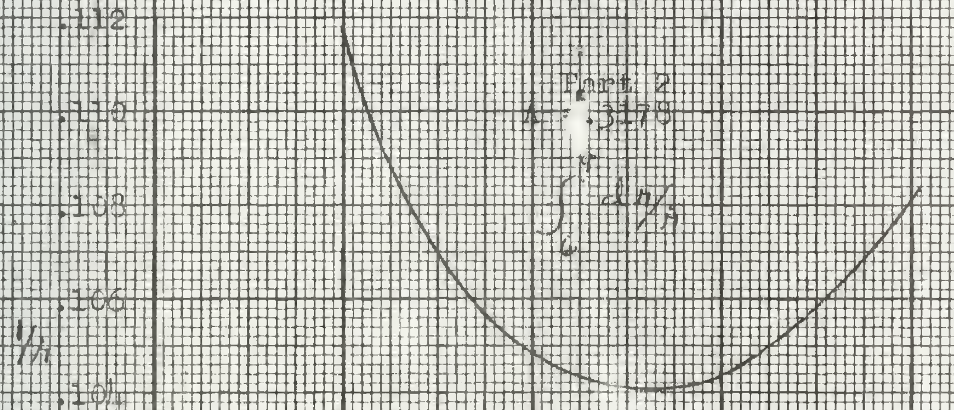

Fig. A17

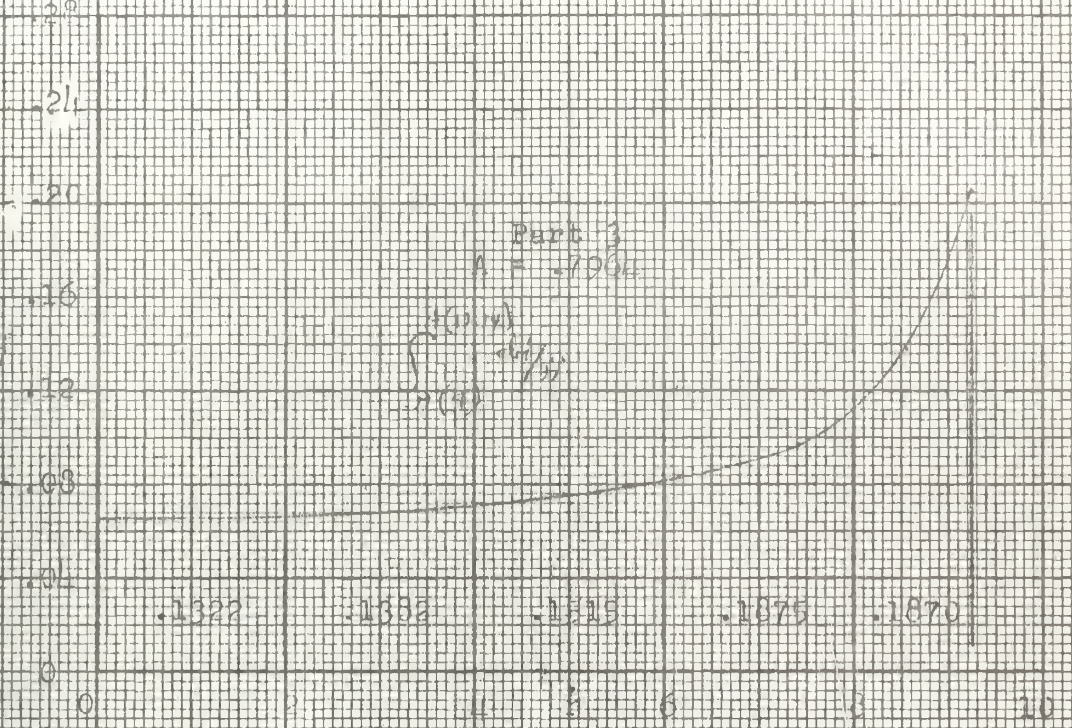

Fig. A18

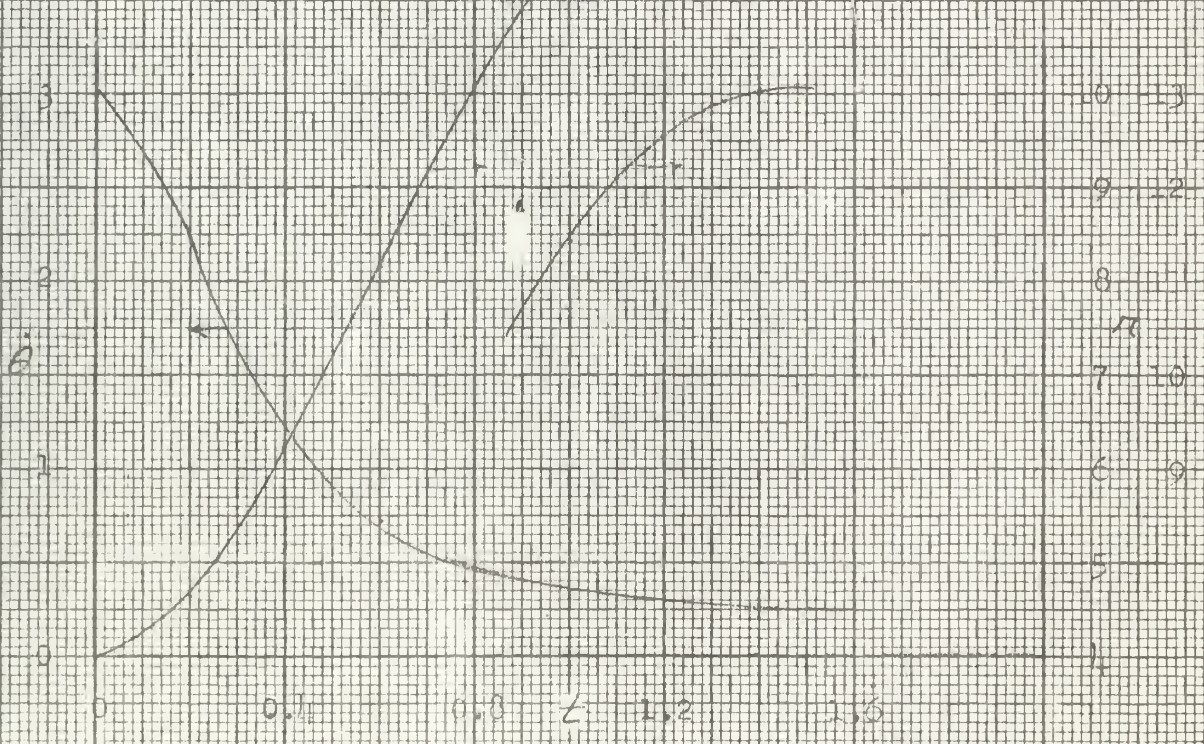

Fig. A19

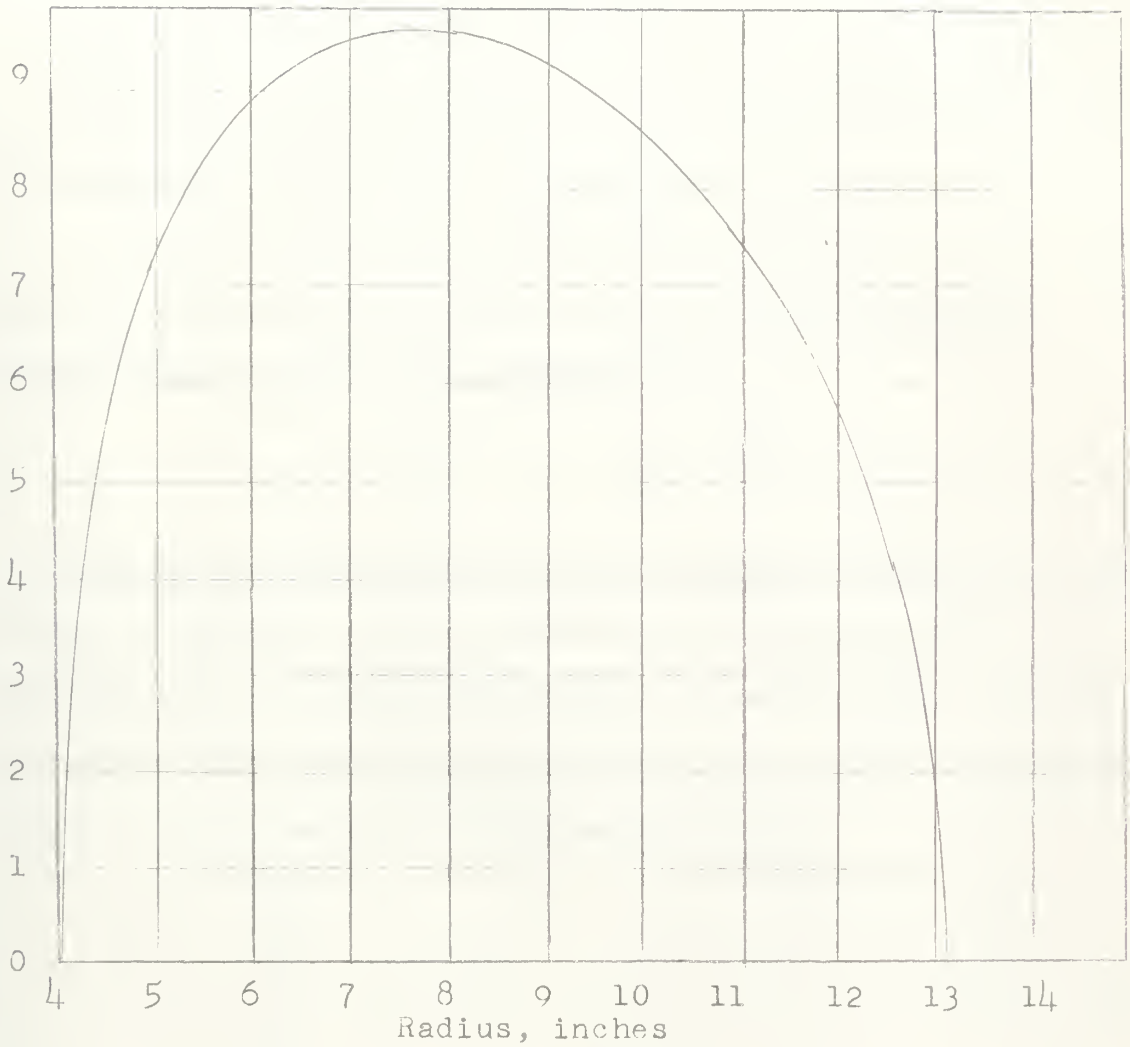

Fig. A20 - r vs radius

APPENDIX II

Scaling Equations for Central Force Problem

The basic equations for this problem are:

$$\ddot{r} = \frac{2304}{r^3} - 2r + 10$$

$$\dot{\theta} = 48/r^2$$

The initial conditions at time $t = 0$ are:

$$r = 4, \quad \dot{r} = 0, \quad \theta = 0, \quad \dot{\theta} = 3,$$

and we also know from the numerical solution that the approximate maximum values of each of the parameters are:

$$r = 13.14 \qquad \dot{r} = 9.6$$

$$\ddot{r} = 38 \qquad \dot{\theta} = 3$$

Also from the numerical solution we know that the time from apse to apse is about 1.46 seconds. For our solution we will select $\theta = 3\pi$, (1.5 revolutions) as we are only interested in the initial aspect. Knowing the maximum values and using the relationship $x = \alpha_X X$, as explained in Sec. 6, we can now solve for the scaling factors.

$$\alpha_r = \frac{13.14}{100} = .1314 \qquad \therefore \text{ Let } \alpha_r = .2 \qquad R_{max} = 65.7V$$

$$\alpha_{\dot{r}} = \frac{9.6}{100} = .096 \qquad \therefore \text{ Let } \alpha_{\dot{r}} = .12 \qquad \dot{R}_{max} = 80V$$

$$\alpha_{\ddot{r}} = \frac{38}{100} = .38 \qquad \therefore \text{ Let } \alpha_{\ddot{r}} = .5 \qquad \ddot{R}_{max} = 76V$$

$$\alpha_\theta = \frac{3\pi}{100} = .0942 \qquad \therefore \text{ Let } \alpha_\theta = .12 \qquad \theta_{max} = 78.5V$$

$$\alpha_{\dot{\theta}} = \frac{3}{100} = .03 \qquad \therefore \text{ Let } \alpha_{\dot{\theta}} = .04 \qquad \dot{\theta}_{max} = 75V$$

For time scaling we wish to slow the problem time down so we assume $t_c = 5t_p$, giving a period for the computer of 7.3 seconds, apse to apse.

Now using the above scaling factors we can proceed to scale the basic equations for the computer:

$$\alpha_{\ddot{r}} \ddot{R} = \frac{2304}{(\alpha_r^3 R^3)} - 2\alpha_r R + 10$$

$$\ddot{R} = \frac{2304}{\alpha_{\ddot{r}} \alpha_r^3 R^3} - \frac{2\alpha_r R}{\alpha_{\ddot{r}}} + \frac{10}{\alpha_{\ddot{r}}}$$

$$= \frac{2304}{.5(.2)^3 R^3} - \frac{2 \cdot 0.2 R}{.5} + \frac{10}{.5}$$

$$= \frac{576,000}{R^3} - .8R + 20$$

$$\alpha_{\dot{\theta}} \dot{\theta} = \frac{48}{(\alpha_r R)^2}$$

$$\dot{\theta} = \frac{48}{\alpha_{\dot{\theta}} \alpha_r^2 R^2}$$

$$= \frac{48}{.04(.2)^2 R^2}$$

$$= \frac{30,000}{R^2}$$

The equations are now in the form $\ddot{R} = -(\omega_1 Z - \omega_2 R + \omega_3 \cdot 20)$ where $Z = 576,000/R^3$, and $\dot{\theta} = 30,000/R^2$. Because Z and $\dot{\theta}$ are developed by the division circuits and 20 is a constant voltage their a_i values (coefficient potentiometer settings) are each 1. Thus their corresponding resistors are all 1M. To determine the value of a_2 we use the relationship $\omega_i = \frac{a_i R_f}{R_i}$. Equating ω_2 to $\frac{a_2 R_{f2}}{R} = .8$ and letting R_2 and R_{f2} equal 1M, we find a_2 to be .8.

To obtain $\dot{R}$ and R we must integrate $\int \ddot{R}\, dt_p$ and $\int \dot{R}\, dt_p$.
We therefore scale these equations as follows:

$$\alpha_{\dot{R}} \dot{R} = -\int \frac{\alpha_{\ddot{R}} \ddot{R}}{\alpha_t} dt_c$$

$$\dot{R} = -\frac{\alpha_{\ddot{R}}}{\alpha_{\dot{R}}\alpha_t} \int \ddot{R}\, dt_c$$

$$= -\frac{.5}{.12 \cdot 5} \int \ddot{R}\, dt_c = -.833 \int \ddot{R}\, dt_c = -\omega_4 \int \ddot{R}\, dt_c$$

$$\alpha_R R = -\int \frac{\alpha_{\dot{R}} \dot{R}}{\alpha_t} dt_c$$

$$= -\frac{\alpha_{\dot{R}}}{\alpha_R \alpha_t} \int \dot{R}\, dt_c$$

$$= -\frac{.12}{.2 \cdot 5} \int \dot{R}\, dt_c = -.12 \int \dot{R}\, dt_c = -\omega_5 \int \dot{R}\, dt_c$$

Now using the relationship $\omega_i = \frac{a_i}{R_i C_f}$ we solve for a_4 and a_5 as shown below. The resulting values are .833 and .48 respectively using the resistors and capacitors shown.

To complete our scaling we must now determine the a_i value for θ. θ is found by integrating $\int \dot{\theta}\, dt_p$. Scaling this equation we find:

$$\alpha_\theta \theta = -\int \frac{\alpha_{\dot{\theta}} \dot{\theta}}{\alpha_t} dt_c$$

$$\theta = -\frac{\alpha_{\dot{\theta}}}{\alpha_\theta \alpha_t} \int \dot{\theta}\, dt_c = -\frac{.04}{.12 \cdot 5} \int \dot{\theta}\, dt_c$$

$$= -.066 \int \dot{\theta}\, dt_c = -\omega_9 \int \dot{\theta}\, dt_c$$

$$\omega_4 = \frac{a_4}{R_4 C_{f3}} = .833 \qquad R_4 = 1M \quad C_{f3} = 1\mu f$$

$$\therefore a_4 = .833$$

$$\omega_5 = \frac{a_5}{R_5 C_{f4}} = .12 \qquad R_5 = 2M \quad C_{f4} = 2\mu f$$

$$\therefore a_5 = .48$$

$$\omega_9 = \frac{a_9}{R_9 C_{f5}} = .066 \qquad R_9 = 5M \quad C_{f5} = 1\mu f$$

$$\therefore a_9 = .33$$

The problem is now ready for the computer. The table on the following page summarizes the values for all resistors and capacitors used for this problem.

Table of Circuit Elements		
Amplifier (See Fig. 3)	Circuit Element and Value	Remarks
1	$R_{71} = 1M$ $R_{72} = 10M$	Forms $-Z$
2	$R_1 = 1M$ $R_2 = 1M$ $R_3 = 1M$ $R_{f2} = 1M$ $a_1 = 1.0$ $a_2 = 0.8$ $a_3 = 1.0$	Sums $+\ddot{R}$
3	$R_4 = 1M$ $C_{f3} = 1\mu f$ $a_4 = 0.833$	$-\int +\ddot{R}\, dt_p = -\dot{R}$
4	$R_5 = 2M$ $C_{f4} = 2\mu f$ $a_5 = 0.48$	$-\int -R\, dt_p = +R$
5	$R_9 = 5M$ $C_{f5} = 1\mu f$ $a_9 = 0.33$	$-\int -\dot{\theta}\, dt_p = \theta$
7	$R_{81} = 1M$ $R_{82} = 10M$	Forms $-\dot{\theta}$
9	$R_6 = 0.1M$ $R_{f9} = 0.2M$ $a_6 = 1.0$	$2K^2R^3$

Fig. A21

APPENDIX III

The Spinning Top

During this investigation, some attention was also directed to the problem of the spinning top; that is a symmetrical top with one point fixed under the action of gravity. Using the notation of Synge and Griffith, (7), the equation $\dot{x}^2 = 1/A\left(2E - \frac{\beta^2}{C} - 2mgax\right)(1-x^2) - \frac{(\alpha - \beta x)^2}{A^2}$ was reduced to the form of $\dot{x}^2 = Ax^3 - Bx^2 + Cx + D$.

A sample problem was then selected from Applied Mechanics by N. C. Riggs, (8), which consisted of a gyroscope being released with θ equal to 60 degrees and the subsequent motion being an oscillation between 60 and 82 degrees while precessing at a variable rate.

With this problem, as with the central force problem, the powers of x could be obtained using function multipliers. The key to the solution however consists of developing the square root of $\dot{X}^2$. This should be possible using either a function generator or a division circuit similar to that of the central force problem. Using the function generator was tried first. A curve of Y = 10 X 1/2, X and Y being arbitrary voltages from zero to 100, was calculated and set into the function generator. It was then found on scaling the problem that to prevent the term CX from exceeding 100 volts it was necessary to assign to $\dot{X}^2$ a maximum voltage of approximately nine volts. Then when the circuit was assembled it was found that this voltage was too small for the function generator to operate.

When the use of the function generator proved to be unsatisfactory the use of a division circuit was attempted. To use this it was assumed

that the following relationship was valid. We know that for this circuit $Z = \frac{100 R_1 X}{R_2 Y}$. Now letting Y equal Z and then solving for X we find that $X = \frac{R_2 Z^2}{100 R_1}$, or Z equals $10 X^{1/2}$, with $R_1 = R_2 = 1M$. This arrangement was then put into the circuit and computations attempted. It was found that the operational amplifier generating X went from its initial value through zero and then overloaded, i.e., the voltage X (representing $x = \cos\theta$), did not vary as it should since the circuit was unstable.

After the above attempts failed it was decided that this particular problem could not be scaled satisfactorily for an acceptable analog computer solution within the time available.